

# Homomorphisms of modules associated with polynomial matrices with infinite elementary divisors

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<sup>1</sup>The research of the first author was supported by Deutscher Akademischer Austauschdienst under Award No. A/98/25636.

### **Abstract**

If the inverse of a nonsingular polynomial matrix  $L$  has a polynomial part then one can associate with  $L$  a module over the ring of proper rational functions, which is related to the structure of  $L$  at infinity. In this paper we characterize homomorphisms of such modules.

**Mathematical Subject Classifications (2010):** 15B33, 13C12 93B25

**Keywords:** polynomial matrices, proper rational functions, module homomorphisms, duality, infinite elementary divisors, coprimeness

# 1 Introduction

According to Rosenbrock [6] a transfer matrix  $G \in K^{m \times p}$  of rational functions over a field  $K$  admits a *generalized state space realization*

$$G(s) = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} sI - A_1 & 0 \\ 0 & sN_2 - I \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

such that

$$G_1(s) = C_1(sI - A_1)^{-1}B_1 \quad (1.1)$$

is the strictly proper part and

$$G_2(s) = C_2(sN_2 - I)^{-1}B_2, \quad (1.2)$$

where  $N_2$  is nilpotent, is the polynomial part of  $G$ . It is well known that the realizations (1.1) and (1.2) can be constructed by module theoretic approaches. In the case of (1.1) a construction is due to Fuhrmann [2]. For a realization theory of anticausal input output maps we refer to Conte and Perdon [1]. To describe the polynomial models that serve as state spaces for (1.1) and (1.2) we use the following notation. A rational function  $f \in K(s)$  is called *proper* or *causal* (resp. *strictly proper* or *strictly causal*) if  $f = 0$  or if  $f \neq 0$  and  $f = p/q$ ,  $p, q \in K[s]$ ,  $q \neq 0$ , and  $\deg p \leq \deg q$  (resp.  $\deg p < \deg q$ ). Let  $K_\infty(s)$  denote the ring of proper rational functions over  $K$ . Then

$$K(s) = K[s] \oplus s^{-1}K_\infty(s). \quad (1.3)$$

To (1.3) correspond projection operators

$$\pi_- : K(s) \rightarrow s^{-1}K_\infty(s)$$

and

$$\pi_+ = (I - \pi_-) : K(s) \rightarrow K[s].$$

Put

$$(f)_0 = (\pi_+ f)(0), \quad f \in K(s). \quad (1.4)$$

The decomposition (1.3), the projections  $\pi_-$  and  $\pi_+$ , and definition (1.4) extend naturally from  $K(s)$  to  $K^n(s)$  and  $K^{m \times p}(s)$ .

Let  $G \in K^{m \times p}(s)$  have a realization

$$G = W_1 + P_1 D_1^{-1} Q_1 \quad (1.5)$$

where  $W_1, P_1, Q_1, D_1$  are polynomial matrices, with  $D_1$  of size  $n_1 \times n_1$ . In Fuhrmann's theory [4] a state space for a realization (1.1) of  $\pi_- G$  is provided by

$$V_{D_1} = K_1^n[s] / D_1 K_1^n[s].$$

Obviously  $V_{D_1}$  is a  $K[s]$ -module and therefore also a vector space over  $K$ . The counterpart of (1.5) is a realization

$$G = W_2 + P_2 D_2^{-1} Q_2, \quad (1.6)$$

where  $P_2$  and  $Q_2$  are proper rational matrices,  $W_2$  is strictly proper rational and  $D_2$  is a polynomial matrix,  $D_2 \in K^{n_2 \times n_2}$ . Define

$$U^{D_2} = K_\infty^{n_2}(s) / (K_\infty^{n_2}(s) \cap D_2 s^{-1} K_\infty^{n_2}(s)). \quad (1.7)$$

Then  $U^{D_2}$  is a  $K_\infty(s)$ -module and at the same time a  $K$ -vector space. At the end of this section we shall indicate why  $U^{D_2}$  can be taken as a state space of a realization (1.2) of  $\pi_+ G$ . Let us mention that the finite and infinite *pole modules* (see [9]) of  $G(s)$  are given by  $V_{D_1}$  and  $U^{D_2}$ , if (1.5) is an irreducible realization and (1.6) satisfies coprimeness conditions of the form (3.14).

We note that a nonsingular polynomial matrix  $L \in K^{n \times n}[s]$  gives rise to two types of modules, namely the  $K[s]$ -module

$$V_L = K^n[s] / LK^n[s]$$

and the  $K_\infty(s)$ -module

$$U^L = K_\infty^n(s) / (K_\infty^n(s) \cap Ls^{-1} K_\infty^n(s)). \quad (1.8)$$

Beside realizations there is a wide range of issues such as similarity of state space models, system equivalence or simulation of restricted input output maps which involve two polynomial matrices  $L$  and  $L_1$  and homomorphisms from  $V_L$  to  $V_{L_1}$  and from  $U^L$  to  $U^{L_1}$ . The  $K[s]$ -module homomorphisms from  $V_L$  to  $V_{L_1}$  are well understood. According to Fuhrmann [4] their description is based on intertwining relations between  $L$  and  $L_1$ . In this note we study  $K_\infty(s)$ -module homomorphisms from  $U^L$  to  $U^{L_1}$ . Our characterizations will be in correspondance with Fuhrmann's results in Ref. [2, 4]. Comparing the definitions of  $V_L$  and  $U^L$  we observe that  $LK^n[s]$  is a submodule of  $K^n[s]$  whereas in general  $Ls^{-1} K_\infty^n(s)$  is not contained in  $K_\infty^n(s)$ . Hence it is not surprising that  $U^L$  is less easy to handle than  $V_L$  and that in our study technical obstacles have to be removed which do not appear in the case of the module  $V_L$ .

To obtain a concrete representation of  $U^L$  we define a map

$$\rho^L : K_\infty^n(s) \rightarrow K^n[s]$$

by

$$\rho^L x = L\pi_+ L^{-1} x, \quad x \in K_\infty^n(s).$$

Put  $\bar{x} = \rho^L x$ . For  $q \in K_\infty(s)$  and  $\bar{x} \in \text{Im } \rho^L$  we set  $q \cdot \bar{x} = \overline{qx}$ . This product is well defined since

$$\text{Ker } \rho^L = (K_\infty^n(s) \cap s^{-1} L K_\infty^n(s)).$$

Therefore  $\text{Im } \rho^L$  is a  $K_\infty(s)$ -module, isomorphic to the quotient module  $U^L$  in (1.8). From now on we identify both modules such that

$$U^L = \text{Im } \rho^L = L\pi_+ L^{-1} K_\infty^n(s).$$

Clearly,  $U^L = 0$  if  $sL^{-1}$  is proper rational. A shift operator  $S_-(L)$  on  $U^L$  is given by

$$S_-(L)\bar{x} = s^{-1} \cdot \bar{x}, \quad \bar{x} \in U^L.$$

Clearly,  $S_-(L)$  is a nilpotent endomorphism of  $U^L$ .

Let us now give a concrete example for the use of  $K_\infty(s)$ -module  $U^L$ . Based on the representation (1.6) of  $G$  we derive a realization of  $\pi_+ G$  having  $U^{D_2}$  as its state space. We adapt a construction of [3]. Assume  $\pi_+ G(s) = \sum_{\nu=0}^t G_\nu s^\nu$ . Define the map  $B_2 : K^p \rightarrow U^{D_2}$  by

$$B_2 \xi = \rho^{D_2} Q_2 \xi, \quad \xi \in K^p.$$

Put  $N_2 = S_-(D_2)$  and define  $C_2 : U^{D_2} \rightarrow K^m$  by

$$C_2 \bar{x} = - (P_2 D_2^{-1} \bar{x})_0, \quad \bar{x} \in U^{D_2}.$$

Then a straightforward computation yields

$$G_\nu = -C_2 N_2^\nu B_2, \quad \nu = 0, 1, \dots, t,$$

such that

$$\sum_{\nu=0}^t G_\nu s^\nu = C_2 (sN_2 - I)^{-1} B_2.$$

## 2 Basic facts of the module $U^L$

For a nonzero proper rational function  $f = p/q$ ,  $p, q \in K[s]$ , let a degree function be defined by  $\delta(p/q) = \deg q - \deg p$ . It is well known that  $(K_\infty(s), \delta)$  is a euclidean domain. The units  $K_\infty^*(s)$  are the proper rational functions  $f$  with  $\delta f = 0$ . The ideal  $(s^{-1})$  is the unique maximal ideal of  $K_\infty(s)$ . Let us call a matrix  $P \in K_\infty^{n \times n}(s)$  *bicausal* if  $\det P \in K_\infty^*(s)$ , i.e. if  $P$  is invertible in  $K_\infty^{n \times n}(s)$ . If  $W \in K^{m \times r}(s)$  has rank  $n$  then there exist bicausal matrices  $P$  and  $Q$  such that

$$W = P \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} Q$$

with

$$\Sigma = \text{diag}(s^{-\alpha_1}, \dots, s^{-\alpha_t}, s^{\beta_{t+1}}, \dots, s^{\beta_n}),$$

$$-\alpha_1 \leq \dots \leq -\alpha_t < 0 \leq \beta_{t+1} \leq \dots \leq \beta_n. \quad (2.1)$$

The integers  $-\alpha_1, \dots, \beta_n$  are uniquely determined by  $W$ . In particular, if  $L \in K^{n \times n}[s]$  is nonsingular then

$$s^{-1}L = P\Sigma Q \quad (2.2)$$

for some  $P, Q \in K_\infty^{n \times n}(s)^*$  and  $\Sigma$  as in (2.1). In the case of a linear pencil  $L(s) = A_0 - A_1 s$  the polynomials  $s^{\alpha_1}, \dots, s^{\alpha_t}$  are the elementary divisors of  $A_0 s - A_1$  belonging to the characteristic root 0. According to [7] the matrix  $\Sigma$  in (2.2) and (2.1) provides information on the structure of  $U^L$ . We have

$$U^L \cong \oplus \{K_\infty(s)/s^{-\alpha_j}K_\infty(s), j = 1, \dots, t\}$$

such that  $U^L$  is a finitely generated torsion module over  $K_\infty(s)$  with elementary divisors

$$s^{-\alpha_1}, \dots, s^{-\alpha_t}. \quad (2.3)$$

We call (2.3) the *infinite elementary divisors* of  $L$ . Then  $s^{\alpha_1}, \dots, s^{\alpha_t}$  are the elementary divisors of the shift  $S_-(L)$ , and  $\dim_K U^L = \alpha_1 + \dots + \alpha_t$ . To describe a dual pairing [8] between the  $K$ -linear spaces  $U^{L^T}$  and  $U^L$  we note that

$$\langle \bar{y}, \bar{x} \rangle = (y^T L^{-1} x)_0, \quad \bar{y} \in U^{L^T}, \bar{x} \in U^L, \quad (2.4)$$

is a well defined nondegenerate bilinear form on  $U^{L^T} \times U^L$ .

### 3 Homomorphisms

Our main result is Theorem 3.3 below. Its proof will be based on the subsequent two lemmas. In the following  $L \in K_\infty^{n \times n}(s)$  and  $L_1 \in K_\infty^{n_1 \times n_1}(s)$  will be fixed nonsingular polynomial matrices.

**Lemma 3.1.** *A map*

$$\Phi : K_\infty^n(s) \rightarrow U^{L_1} \quad (3.1)$$

*is a  $K_\infty(s)$ -module homomorphism if and only if there exists a matrix  $\Theta \in K_\infty^{n_1 \times n}(s)$  such that*

$$\Phi x = \rho^{L_1}(\Theta x), \quad x \in K_\infty^n(s). \quad (3.2)$$

*Proof.* Let  $e_1, \dots, e_n$  be the standard basis of  $K^n$ . Assume that  $\Phi$  in (3.1) is a  $K_\infty(s)$ -module homomorphism. Then  $\Phi e_i = \rho^{L_1} \theta_i$  for some  $\theta_i \in K_\infty^{n_1}(s)$  and (3.2) holds with  $\Theta = (\theta_1, \dots, \theta_n)$ . The converse is obvious. ■

Condition (3.3) below together with a somewhat technical equivalent condition will be crucial.

**Lemma 3.2.** *We have*

$$\Theta \operatorname{Ker} \rho^L \subseteq \operatorname{Ker} \rho^{L_1}. \quad (3.3)$$

*with  $\Theta \in K_\infty^{n_1 \times n}(s)$  if and only if there exist a matrix  $\Theta_1 \in K_\infty^{n_1 \times n}(s)$  and a matrix  $\Psi$  satisfying*

$$\Psi \in s^{-1} K_\infty^{n_1 \times n}(s) \text{ and } L_1 \Psi \in K_\infty^{n_1 \times n}(s) \quad (3.4)$$

*such that*

$$(\Theta + L_1 \Psi)L = L_1 \Theta_1. \quad (3.5)$$

*Proof.* It is evident that (3.5) implies (3.3). To prove the converse implication we note that (3.3) is equivalent to  $\Theta \operatorname{Ker} \rho^L \subseteq s^{-1} L_1 K_\infty^{n_1}(s)$ . If  $s^{-1} L$  is factorized as in (2.2),

$$\begin{aligned} s^{-1} L &= P \Sigma Q, \quad \Sigma = \operatorname{diag}(A, B), \\ A &= \operatorname{diag}(s^{-\alpha_1}, \dots, s^{-\alpha_t}), \quad B = \operatorname{diag}(s^{\beta_{t+1}}, \dots, s^{\beta_n}) \end{aligned} \quad (3.6)$$

then  $\operatorname{Ker} \rho^L = P \operatorname{diag}(A, I) K_\infty^n(s)$ . Hence if

$$G = L_1^{-1} \Theta P \operatorname{diag}(A, I)$$

then (3.3) is equivalent to  $G \in s^{-1} K_\infty^{n_1 \times n}(s)$ . From (3.6) and

$$\Sigma = \operatorname{diag}(A, 0) + \operatorname{diag}(0, B)$$

we obtain

$$L_1^{-1} \Theta L = G \operatorname{diag}(I, 0) Q + L_1^{-1} \Theta P \operatorname{diag}(0, I) P^{-1} L.$$

Now choose

$$\Psi = -G \operatorname{diag}(I, 0) Q.$$

Then  $\Psi$  satisfies (3.4) and if we put  $\Theta_1 = L_1^{-1} \Theta L + \Psi L$  then we have  $\Theta_1 \in K_\infty^{n_1 \times n}(s)$ , which proves (3.5). ■

We extend the map  $\rho^{L_1}$  to  $K^n(s)$  and define

$$\rho_e^{L_1} = L_1 \pi_+ L_1^{-1} w, \quad w \in K^n(s).$$

**Theorem 3.3.** *The map  $\phi : U^L \rightarrow U^{L_1}$  is a  $K_\infty(s)$ -module homomorphism if and only if there exist matrices  $\Theta, \Theta_1 \in K_\infty^{n_1 \times n}(s)$  such that*

$$\Theta L = L_1 \Theta_1 \quad (3.7)$$

and

$$\phi \bar{x} = \rho_e^{L_1} \Theta \bar{x}, \quad \bar{x} \in U^L. \quad (3.8)$$

If (3.7) holds then we have

$$\rho_e^{L_1} \Theta \bar{x} = \rho_e^{L_1} \Theta x \quad (3.9)$$

for all  $x \in K_\infty^n(s)$ .

*Proof.* Let us show first that (3.7) implies (3.9). We have

$$\begin{aligned} \rho_e^{L_1} \Theta \bar{x} &= L_1 \pi_+ L_1^{-1} \Theta \bar{x} = L_1 \pi_+ \Theta_1 L^{-1} \bar{x} = \\ &= L_1 \pi_+ \Theta_1 L^{-1} x = L_1 \pi_+ L_1^{-1} \Theta x = \rho_e^{L_1} \Theta x. \end{aligned} \quad (3.10)$$

Now let  $\phi : U^L \rightarrow U^{L_1}$  be a  $K_\infty(s)$ -module homomorphism. Define  $\Phi = \phi \rho^L$  such that

$$\Phi x = \phi \bar{x}, \quad x \in K_\infty^n(s). \quad (3.11)$$

Then  $\Phi : K_\infty^n(s) \rightarrow U^{L_1}$  is also a  $K_\infty(s)$ -module homomorphism. Because due to Lemma 3.1 there exists a  $\tilde{\Theta} \in K_\infty^{n_1 \times n}(s)$  such that

$$\Phi x = \rho^{L_1} \tilde{\Theta} x. \quad (3.12)$$

It follows from (3.11) that  $x, v \in K_\infty^n(s)$  and  $\bar{x} = \bar{v}$  imply  $\rho^{L_1} \tilde{\Theta} x = \rho^{L_1} \tilde{\Theta} v$ . Therefore we obtain

$$\tilde{\Theta} \text{Ker } \rho^L \subseteq \text{Ker } \rho^{L_1}. \quad (3.13)$$

We can replace  $\tilde{\Theta}$  in (3.12) and (3.13) by  $\Theta = \tilde{\Theta} + L_1 \Psi$  if  $\Psi \in s^{-1} K_\infty^{n_1 \times n}(s)$  and  $L_1 \Psi \in K_\infty^{n_1 \times n}(s)$ . From Lemma 3.2 we know that starting from (3.13) we can find a  $\Psi$  which yields (3.7) with  $\Theta_1 \in K_\infty^{n_1 \times n}(s)$ . Thus we have shown that

$$\phi \bar{x} = \rho^{L_1} \Theta x = \rho_e^{L_1} \Theta \bar{x}$$

with  $\Theta$  satisfying a relation (3.7).

Conversely, if a map  $\phi : U^L \rightarrow U^{L_1}$  is defined by (3.7) and (3.8) then it is easy to verify that  $\phi$  is a  $K_\infty(s)$ -module homomorphism.  $\blacksquare$

We remark that Theorem 3.3 remains true if condition (3.7) is replaced by

$$\pi_+ L_1^{-1} \Theta = \pi_+ \Theta_1 L^{-1}.$$

Given the duality (2.4) between  $U^L$  and  $U^{L^T}$  it is not difficult to obtain the dual map of  $\phi$ . We set  $\bar{w} = \rho^{L^T} w$ ,  $w \in K_\infty^{n_1}(s)$ .



**Theorem 3.4.** Let  $\Theta, \Theta_1 \in K_\infty^{n_1 \times n}(s)$  be such that  $\Theta L = L_1 \Theta_1$ . Let  $\phi : U^L \rightarrow U^{L_1}$  be defined by (3.8). Then the dual map

$$\phi^* : U^{L_1^T} \rightarrow U^{L^T}$$

is given by

$$\phi^* \bar{w} = \rho^{L^T} \Theta_1^T w, \quad \bar{w} \in U^{L_1^T}.$$

We now turn to surjectivity and injectivity. For a pair  $\Theta \in K_\infty^{n_1 \times n}(s)$  and  $L_1 \in K^{n_1 \times n_1}$  we set  $(\Theta, s^{-1}L_1)_l = I$  if there exist proper rational matrices  $C$  and  $D$  such that

$$\Theta C + s^{-1}L_1 D = I. \quad (3.14)$$

Similarly, for  $\Theta_1 \in K_\infty^{n_1 \times n}(s)$  and  $L \in K^{n \times n}$  we write  $(\Theta_1, s^{-1}L)_r = I$  if  $(\Theta_1^T, s^{-1}L^T)_l = I$ .

**Theorem 3.5.** Let  $\phi : U^L \rightarrow U^{L_1}$  be defined by (3.9) and (3.7). Then

- (i)  $\phi$  is surjective if and only if  $(\Theta, s^{-1}L_1)_l = I$ ,
- (ii)  $\phi$  is injective if and only if  $(\Theta_1, s^{-1}L)_r = I$ .

*Proof.* (i) Assume first that  $\phi$  is surjective. Let  $w \in K_\infty^{n_1}(s)$  be given. Then  $\rho^{L_1} w = \rho^{L_1} \Theta v$  for some  $v \in K_\infty^n(s)$ . We have  $w - \Theta v \in \text{Ker } \rho^{L_1}$ , which implies

$$w \in \Theta K_\infty^n(s) + s^{-1}L_1 K_\infty^n(s)$$

or equivalently  $(\Theta, s^{-1}L_1)_l = I$ . Conversely, suppose that (3.14) holds. To show that  $w = \rho^{L_1} x$  is in  $\phi U^L$  we note that (3.14) implies  $x = \Theta v + s^{-1}L_1 x_2$  for some  $v \in K_\infty^n(s)$ ,  $x_2 \in K_\infty^{n_1}(s)$ . Because of  $s^{-1}L_1 x_2 \in \text{Ker } \rho^{L_1}$  we obtain  $w = \rho^{L_1} \Theta v = \phi \bar{v}$ .

(ii) By duality the statement follows at once from (i). ■

If  $M$  is a finitely generated  $p$ -module over a principal ideal domain and  $S$  is a submodule and  $Q$  is a quotient module of  $M$  then the relations between the invariants of  $M$  and those of  $S$  and  $Q$  are well known (see e.g. [5, p. 92, 93]). We complete our note with a corresponding observation on the existence of surjective and injective homomorphisms. Let

$$s^{-\alpha_1}, \dots, s^{-\alpha_t}, \quad \alpha_1 \geq \dots \geq \alpha_t,$$

and

$$s^{-\gamma_1}, \dots, s^{-\gamma_p}, \quad \gamma_1 \geq \dots \geq \gamma_p,$$

be the infinite elementary divisors of  $L$  and  $L_1$ , respectively. Then there exists a surjective  $K_\infty(s)$ -module homomorphism  $\phi : U^L \rightarrow U^{L_1}$  if and only if

$$t \geq p \quad \text{and} \quad \alpha_1 \geq \gamma_1, \dots, \alpha_p \geq \gamma_p,$$

and there exists an injective  $\phi$  if and only if

$$t \leq p \quad \text{and} \quad \alpha_1 \leq \gamma_1, \dots, \alpha_t \leq \gamma_t.$$

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